

Substitution equation of complex quaternion with variable can reduce one variable

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Abstract. It is well known that the advantage of complex number over real number is that it can express vector, while real number can only express scalar. two real numbers can describe two-dimensional coordinates, and a complex number can describe two-dimensional coordinates; three real numbers can describe three-dimensional coordinates, and four real numbers can describe four-dimensional coordinates, but a quaternion can describe two-dimensional, three-dimensional or four-dimensional coordinates, so a complex or quaternion containing variables can reduce an independent variable when replacing the equation. In this paper, the complex number containing variables is used to describe a circle, a two-dimensional circle, and a quaternion containing variables is used to describe a three-dimensional the surface of the ball and a three-dimensional sphere, using quaternion with variables to describe four-dimensional the surface of the ball and four-dimensional sphere . It is also found that the complex number containing a variable can describe the plane golden spiral and the quaternion containing a variable can describe three-dimensional golden spiral. It is also found that if $a^b=a^c$ can not deduce $b=c$ when the exponent is a complex number.

Key words: Complex number; quaternion; circle; round surface; three dimensional sphere; four dimensional sphere; golden ratio; golden spiral.

1. Introductory topic

Compared with real numbers, the superiority of complex numbers is that they can represent vectors, while real numbers can only represent scalars. In addition, a single real number can only describe a one-dimensional coordinate, two real numbers are required to describe a two-dimensional coordinate, three real numbers for a three-dimensional coordinate, and four real numbers for a four-dimensional coordinate. However, a complex number can describe a two-dimensional coordinate, and a quaternion can describe a one-dimensional, two-dimensional, three-dimensional, or four-dimensional coordinate. Therefore, when a complex number or quaternion^[1] containing variables is used to replace an equation, one variable can be reduced.

2. Geometric Meaning and Wonderful Applications of the Complex Number a^i

2.1 a^i represents a circle with a radius of R (a is a variable and $a > 0$)

According to Euler's formula $e^{ix}=\cos x+i\sin x$ ^[2] we know that: $a^i=e^{i\ln(a)}=\cos(\ln a)+i\sin(\ln a)$ ($a \in R, a > 0$). The standard equation of a circle is $x^2+y^2=R^2$, and $[\cos(\ln a)]^2+[\sin(\ln a)]^2=1$, which just satisfies the equation of a circle with a radius of 1. This indicates that a^i must be on a circle with a unit radius of 1. If a is a variable, when a changes within a certain range of values, a^i represents a circle with a unit radius of 1.

When does a change within what range of values so that a^i represents a complete circle with a unit radius of 1? When $a=1$, $\ln a=0$, then $a^i=\cos 0+i\sin 0=1$. Taking this as the starting point, when $a=e^{\pi/4} \approx 2.19$, then $a^i=\cos \pi/4+i\sin \pi/4 =\sqrt{2}/2+i\sqrt{2}/2$. When a gradually increases to $e^{2\pi} \approx 535.49$, then $a^i=\cos 2\pi+i\sin 2\pi=1$, which shows that $1^i=(e^{2\pi})^i$, that is, when a changes between 1 and $e^{2\pi}$, a^i just describes a circle. If it continues to change to $e^{4\pi} \approx 286751.31$, $a^i=1$, and it goes through another circle. Continuing to change to $e^{6\pi}, e^{8\pi}, \dots$ that is, every time a changes by $e^{2\pi}$, a^i will go through a complete circle, and it shows that when $a > 1$, a^i goes through complete circles

more and more slowly as a increases!

Then when $a < 1$, does a^i also go through some circles? For example, when $a = e^{-2\pi} \approx 0.0018674$, $a^i = \cos(-2\pi) + i\sin(-2\pi) = 1$. When a changes to $e^{-4\pi} \approx 0.0000034873$, $a^i = 1$, and when it changes to $e^{-6\pi}, e^{-8\pi}, \dots$, $a^i = 1$, that is, when $0 < a < 1$, a^i goes through complete circles faster and faster as a decreases!

a^i (a is a variable and $a > 0$) represents a circle with a unit radius of 1, then Ra^i (a is a variable and $a > 0$) represents a circle with a radius of R . If both R and a are variables, then Ra^i represents a circular surface or an annular surface.

2.2 $a^{450000137857+i}$ represents the planar golden spiral

$a^{1+i} = aa^i$. If a keeps getting larger, then a^{1+i} no longer represents a circle, but a planar spiral with an increasingly larger polar radius. The polar coordinate equation of the Archimedean spiral is $r = a + b\theta$, where $b = dr/d\theta$ contains two variables, while a^{1+i} contains only one variable.

Take several special values of a and look at its coordinates on the spiral trajectory: When $a = 1$, $a^{1+i} = 1$. When $a = e^{4\pi} \approx 2.19$, then $a^{1+i} = e^{\pi/4} (\sqrt{2}/2 + i\sqrt{2}/2)$, When $a = e^{2\pi} \approx 535$, $a^{1+i} = e^{2\pi} \approx 535$. When $a = e^{4\pi} \approx 286751$, $a^{1+i} = e^{4\pi} \approx 286751$. To form a spiral with the same number of turns, the change of a is not uniform, and the radius of the spiral increases more and more rapidly as the number of turns increases. Such a spiral has no aesthetic feeling and will be rarely used in daily life, for example, in the design of industrial products, landscape architecture design, etc.

The geometric relationship that satisfies the golden ratio φ (The golden ratio refers to dividing the whole into two parts, and the ratio of the whole part to the larger part is equal to the ratio of the larger part to the smaller part, and its ratio is approximately $\varphi = 1.6180339$) is generally recognized as the most aesthetically pleasing proportion. If the spiral can grow in a golden ratio pattern, it will present a kind of mathematical beauty. Can we find a set of changing complex numbers to describe the golden spiral?

$a^{b+i} = a^b a^i$, Since a^i represents a circle with a unit radius of 1, the value of b determines the size of the polar radius of the spiral. Taking an appropriate value of b , a^{b+i} can replace the equation of the golden spiral.

For the complex number $a^{450000137857+i}$, when a changes, its coordinate values expand along the spiral. Every time it rotates by 90 degrees, the polar radius increases by approximately 1.6180339 times, that is, it just satisfies the growth according to the golden ratio φ , and it is a planar golden spiral.

Verify with special values: If $a = 1$, $a^{450000137857+i} = 1$,
If $a = e^{\pi/2} \approx 4.81$, $a^{450000137857+i} = e^{450000 \times 2137857\pi} (\cos 2\pi + i\sin 2\pi) \approx 1.6180339i = \varphi i$, The two points differ by 90 degrees, and the ratio of the polar radii is φ ; If $a = e^{\pi} \approx 23.14$, $a^{450000137857+i} = -\varphi^2$;
If $a = e^{3\pi/2} \approx 111.32$, $a^{450000137857+i} = \varphi^3 i$. If $a = e^{2\pi}$, $a^{450000137857+i} = \varphi^4$;
If $a = e^{5\pi/2}$, $a^{450000137857+i} = \varphi^5 i, \dots$. Every time it rotates by 90 degrees, the polar radius increases by φ times.

If $a = e^{-2\pi}$, $a^{450000137857+i} = -\varphi^{-1} i$. If $a = e^{-\pi}$, $a^{450000137857+i} = -\varphi^{-2}$.
If $a = e^{-3\pi/2}$, $a^{450000137857+i} = \varphi^{-3} i, \dots$. That is, when the polar radius is less than 1, it still satisfies that every time it rotates by 90 degrees, the polar radius increases by φ times.

Verify with arbitrary values: If $a = e^{4\pi}$, $a^{137857/450000+i} = e^{137857\pi/(450000 \times 4)} (\sqrt{2}/2 + i\sqrt{2}/2)$. If $a = e^{3\pi/4}$ 时, $a^{137857/450000+i} = e^{137857 \times 3\pi/(450000 \times 4)} (-\sqrt{2}/2 + i\sqrt{2}/2)$. The ratio of the polar radii of the two points is $e^{137857\pi/(450000 \times 2)} \approx \varphi$; If $a = e^{\pi/3}$, $a^{137857/450000+i} = e^{137857\pi/(450000 \times 3)} (1/2 + i\sqrt{3}/2)$; If $a = e^{5\pi/6}$ 时, $a^{137857/450000+i} = e^{137857 \times 5\pi/(450000 \times 6)} (-\sqrt{3}/2 + i1/2)$. The ratio of the polar radii of the two points is $e^{137857\pi/(450000 \times 2)} \approx \varphi$.

2.3 The Relationship between the Polar Radius, Polar Angle and the Golden Ratio

When $a = e^{\theta}$ is a certain fixed value, the complex number $a^{450000137857+i}$ is just a polar coordinate point on the golden spiral. The distance from it to the origin is the polar radius $r = e^{\theta 450000/137857}$ at

this point on the spiral. Substituting $\theta=0, r=1$. Since the spiral cannot start with a polar radius of 1, when $r<1$, the polar angle of a certain point on the spiral is defined as negative. Let the polar angles of two certain points on the golden spiral be θ_1 and θ_2 respectively, where $\theta_2=\theta_1+2\pi$, and the polar radii are r_1 and r_2 respectively. Then $r_2/r_1=e^{137857\pi/450000 \times 2} = \varphi$, so $(2/\pi)\ln\varphi=137857/450000$, then $r=e^{\theta(2/\pi)\ln\varphi}=\varphi^{2\theta/\pi}$, that is, the relationship between the polar radius, polar angle and the golden ratio on the golden spiral is $r=\varphi^{2\theta/\pi}$.

For example: When $\theta=\pi/2$ 时, $r=\varphi$, When $\theta=\pi/3$ 时, $r=\varphi^{2/3}$, When $\theta=\pi$ 时, $r=\varphi$, When $\theta=7\pi/3$ 时, $r=\varphi^{14/3}$, When $\theta=-\pi/2$ 时, $r=\varphi^{-1}$, When $\theta=-\pi$ 时, $r=\varphi^{-2}$, When $\theta=-5\pi/2$ 时, $r=\varphi^{-5}$ Knowing the polar angle, the polar radius can be quickly calculated.

3. Describing Three-dimensional Spheres and Four-dimensional Spheres with Quaternions Containing Variables

3.1 The Wonderful Application of ai in Three Dimensions

As shown in Figure 1: a^i (a is a variable and $a>0$) represents a circle with a unit radius of 1 in the wx plane, and a^j (a is a variable and $a>0$) represents a circle with a unit radius of 1 in the wy plane, which indicates that a^i rotates 90 degrees counterclockwise towards the positive direction of the y -axis to obtain a^j . If it rotates 180 degrees, it gets a^{-i} , rotates 270 degrees to get a^{-j} , and rotates 360 degrees to get a^i again. Since a is a variable and $a>0$, the two circles a^i and a^j must be on a sphere with a radius of 1, and the planes where they are located are perpendicular to each other.

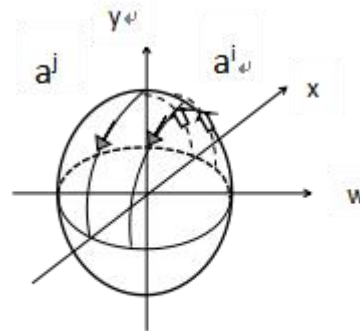


Figure 1: a^i rotates 90 degrees in the positive y -axis direction to obtain a^j

Then can we use a^i and a^j (a is a variable and $a>0$) to describe the entire sphere?

As shown in Figure 2, when a is a variable and $a>0$, a^i represents a circle rotating from the positive direction of the w -axis

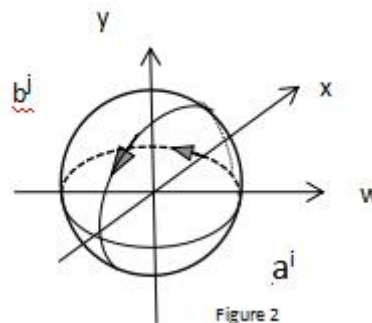
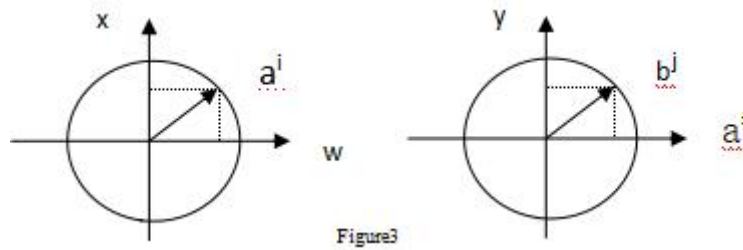


Figure 2

towards the positive direction of the x -axis. Rotating this circle around the positive direction of the y -axis for one week forms a complete sphere. The starting point of the latter circle is a^i , so the

radius of rotation is the cosine of a^i . The latter circle is represented by b^j (b is a variable and $b > 0$). In this way, a^i and b^j (a and b are variables and $a > 0, b > 0$) can describe the entire sphere.



As shown in Figure 3, decompose Figure 2 into two steps. The magnitude of a^i on the w -axis is $\cos(\ln a)$, and on the x -axis is $\sin(\ln a)$. When rotating from a^i (around the x -axis) towards the y -axis, the magnitude of b^j on the w -axis is $\cos(\ln a)\cos(\ln b)$, the magnitude on the x -axis remains unchanged at $\sin(\ln a)$, and on the y -axis is $\cos(\ln a)\sin(\ln b)$. From this, it can be deduced that the spherical surface with a radius of 1 can be described by the quaternion containing two variables a and b , which is $\cos(\ln a)\cos(\ln b) + \sin(\ln a)i + \cos(\ln a)\sin(\ln b)j$ ①.

If $a=1$ and $b=1$, substituting into ①, the coordinates of this point can be calculated as $(1,0,0)$. If $a=e^{\pi/4}$, and $b=1$, the coordinates of this point can be calculated as $(\sqrt{2}/2, \sqrt{2}/2, 0)$. If $a=e^{\pi/4}, b=e^{\pi/4}$, the coordinates of this point are $(1/2, \sqrt{2}/2, 1/2)$. If $a=e^{\pi/4}, b=e^{\pi/3}$, the coordinates of this point are $(\sqrt{2}/4, \sqrt{2}/2, \sqrt{6}/4)$. Substituting the coordinate values into the equation of a sphere $x^2+y^2+z^2=1^2$ with the center at the origin and a radius of 1 for verification, they are all correct. If the radius of this sphere is R , then this sphere is described by $R[\cos(\ln a)\cos(\ln b)+\sin(\ln a)i+\cos(\ln a)\sin(\ln b)j]$ ②. When R is also a variable, it represents a three-dimensional sphere or a hollow sphere.

If the coordinate values of a certain point on the sphere are known, for example, the point is $(1,1,1)$, then $R=\sqrt{3}$ can be calculated. Substituting into ②, $a=e^{\arcsin\sqrt{3}/3}$, $b=e^{\pi/4}$ can be obtained.

3.2 The Ingenious Application of ai in Four Dimensions

Similarly, in four dimensions, when a^i rotates 90 degrees counterclockwise along the positive y -axis, it becomes a^j . Then, when a^j rotates 90 degrees counterclockwise along the positive z -axis, it becomes a^k . When a is a variable and $a > 0$, these three circles a^i , a^j , and a^k must lie on the surface of a four-dimensional sphere with a radius of 1, and the planes they are in are mutually perpendicular. To describe the entire sphere, in the same way, a^i (where a is a variable and $a > 0$) represents a circle on the wx plane. Rotating this circle around the positive y -axis once is represented by b^j (where b is a variable and $b > 0$). Taking this circle on the xy plane and rotating it around the positive z -axis once is represented by c^k (where c is a variable and $c > 0$). Then, this four-dimensional sphere can be described by the equation: $\cos(\ln a)\cos(\ln b)+\sin(\ln a)\cos(\ln c)i+\cos(\ln a)\sin(\ln b)j+ \sin(\ln a)\sin(\ln c)k$ ③

Substituting $a = b = c = e^{\pi/4}$ into equation ③ gives $1/2+1/2i+1/2j+1/2k$, meaning the coordinates of this point are $(1/2, 1/2, 1/2, 1/2)$; substituting $a=e^{\pi/6}, b=e^{\pi/3}, c=e^{\pi/4}$ into equation ③ gives the coordinates of this point as $(\sqrt{3}/4, \sqrt{2}/4, 3/4, \sqrt{2}/4)$.

A four-dimensional sphere with a radius of R can be described by the equation $R[\cos(\ln a)\cos(\ln b)+\sin(\ln a)\cos(\ln c)i+\cos(\ln a)\sin(\ln b)j+ \sin(\ln a)\sin(\ln c)k]$ ④. When R is also a variable, it represents a four-dimensional sphere or a four-dimensional hollow sphere.

3.3 Describing a circle or sphere whose center is not at the origin

The above all discuss the description formula for the center or center of the sphere at the origin. When the center or center of the sphere is not at the origin, it can be regarded as the origin first. This way, the description formula remains unchanged, and then the origin coordinate value can be

directly added. For example, if the center coordinates of the sphere in two dimensions are (x_0, y_0) , then the circle is $a^{i+x_0+iy_0}$. If the semicircle is R, then it is $R(a^{i+x_0+iy_0})$. If the center coordinates of a sphere are (w_0, x_0, y_0) , then the sphere is $[\cos(\ln a)\cos(\ln b)+w_0]+[\sin(\ln a)+x_0]i+[\cos(\ln a)\sin(\ln b)+y_0]j$, the same applies to four-dimensional spheres.

4. Describing a three-dimensional golden spiral with a quaternion containing one variable

If a complex number containing one variable can describe a planar golden spiral, then if not only does it rotate 90 degrees in the plane, increasing the spiral radius by a factor of ϕ , but it also increases in the direction perpendicular to the plane (height) by a factor of ϕ , can it also be described by a number containing variables?

The quaternion $a^{137857/450000+i+a^{137857/450000}j}$ adds a direction, and when a changes, it can describe the three-dimensional golden spiral. Now the value test is:

If $a=1$, $a^{137857/450000+i+a^{137857/450000}j}=1+j$; If $a=e^{\pi/2}$ 时, $a^{137857/450000+i+a^{137857/450000}j}=\phi i+\phi j$; If $a=e^{\pi}$ 时, $a^{137857/450000+i+a^{137857/450000}j}=-\phi^2+\phi^2j$; $a=e^{3\pi/2}$ 时, $a^{137857/450000+i+a^{137857/450000}j}=-\phi^3i+\phi^3j$

If $a=e^{-\pi/2}$ 时, $a^{137857/450000+i+a^{137857/450000}j}=-\phi^{-1}i+\phi^{-1}j$; If $a=e^{-\pi}$ 时, $a^{137857/450000+i+a^{137857/450000}j}=-\phi^{-2}+\phi^{-2}j$; If $a=e^{-3\pi/2}$ 时, $a^{137857/450000+i+a^{137857/450000}j}=\phi^{-3}i+\phi^{-3}j$

The quaternion $a^{137857/450000+i+a^{137857/450000}j}$, which contains one variable, rotates 90 degrees in the plane. This not only increases the helix radius by a factor of ϕ , but also increases the vertical plane direction (height) by a factor of ϕ .

5. Two ways of expressing the tangent point between two circles or two spheres

As shown in Figure 4, there are four circles with half longitude 1 in the four quadrants, with centers of $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$ respectively. They are exactly tangent to the small circle with the center at the origin. The half longitude of the small circle can be calculated as $\sqrt{2}-1$. Therefore, the tangent point of the first quadrant is ra^i small $ra^i_{min}=(2-\sqrt{2})/2+i(2-\sqrt{2})/2$ ($a=e^{\pi/4}$). The large circle can also be used to describe a^i large ($a=e^{5\pi/4}$) 即 $a^i_{max}+x_0+iy_0=(-\sqrt{2}/2+1)+(-\sqrt{2}/2+1)i=(2-\sqrt{2})/2+i(2-\sqrt{2})/2$, which means that the coordinates of the tangent point can be expressed from the angle a^i of the small or large

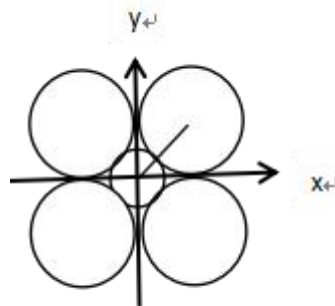


Figure 4

If there are also eight three-dimensional spheres with half longitude 1 in the eight hexagram limits in three-dimensional coordinates, and they are all tangent to the three coordinate axes, and they are tangent to the small ball with the center of the sphere at the origin, the half longitude of the small ball can be calculated as $\sqrt{3}-1$. The first hexagram limit has a large ball center of $(1,1,1)$, and two expressions for the tangent point can also be obtained; In the four-dimensional coordinate system, if there are also eight four-dimensional spheres in the sixteen hexagram limits that are all 1 and a half meridians, and are tangent to the four coordinate axes, they are at the origin with the

center of the sphere $a_{\min}^i = 1/2 + 1/2i + 1/2j + 1/2k$ ($a = e^{\pi/4}$) circle. $a_{\max}^i = w_0 + ix_0 + jy_0 + kz_0$ ($a = e^{5\pi/4}$)
 $= (-1/2 + 1/2i) + (-1/2 + 1/2j) + (-1/2 + 1/2k) = 1/2 + 1/2i + 1/2j + 1/2k$.

6. When the index is a complex number, if $a^b = a^c$, it cannot be deduced that $b=c$.

If a real number is a, then a^i represents a counterclockwise rotation of $\pi/2$, with no change in size. Then $2\pi i$ represents the point reached by rotating $\pi/2$ counterclockwise along a circle with a half longitude of 2π . It represents a vector of size 2π , so $2\pi i \neq 0$; And $e^{2\pi i}$ represents the point reached after rotating counterclockwise by 2π along a unit circle with a half longitude of 1. After rotating counterclockwise by 2π , it just returns to the starting point. It is equivalent to a point rotating once (2π) on the circle and returning to its original position, with the coordinate value unchanged. After rotating $2k\pi$ (k is any integer), it is still equivalent to no rotation. It can be seen that e^z is a periodic function with a period of $2\pi i$, that is, $e^{(z+2k\pi i)} = e^z$ (k is any integer).

In summary, $e^0 = e^{2\pi i}$ cannot be derived as $2\pi i = 0$, which means that when the exponent is complex, if $a^b = a^c$ (even if $a \neq 0$, $a \neq 1$, $a \in \mathbb{R}$, $b \in \mathbb{C}$ or $c \in \mathbb{C}$ or $b, c \in \mathbb{C}$), $b=c$ cannot be deduced.

7. Conclusion

When a is a variable and $a > 0$, a^i represents a circle with a half longitude unit of 1, and Ra^i represents a circle or circular surface; The complex number $a^{137857/450000+i}$ describes the trajectory of a planar golden spiral; When a and b are variables and $a > 0$, $b > 0$, $R[\cos(\ln a)\cos(\ln b) + \sin(\ln a)i + \cos(\ln a)\sin(\ln b)j]$ represents a three-dimensional sphere or hollow sphere; When a, b, and c are variables and $a > 0$, $b > 0$, and $c > 0$, $R[\cos(\ln a)\cos(\ln b) + \sin(\ln a)\cos(\ln c)i + \cos(\ln a)\sin(\ln b)j + \sin(\ln a)\sin(\ln c)k]$ represents a four-dimensional sphere or hollow sphere; The quaternion $a^{137857/450000+i} + a^{137857/450000}j$ containing one variable describes the trajectory of a three-dimensional golden spiral. The relationship between the polar diameter, polar angle, and golden ratio of the golden spiral is $r = \phi^{2\theta/\pi}$; When the index is a complex number, if $a^b = a^c$, it cannot be deduced that $b=c$.

References

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