

Relativistic Local Electrostatic Model and Their Manifestly Covariant Formulations in Plasma Physics

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Abstract. This study introduces a local relativistic electrostatic model with a covariant formulation designed for plasma physics applications. By incorporating relativistic effects and ensuring consistency under Lorentz transformations, the model provides a robust framework for analyzing plasma behaviors across different inertial frames, thereby improving the accuracy of simulations and predictions in high-energy environments.

Keywords: surface plasmon, relativistic effects, local electrostatic, manifestly covariant formulations, plasma physics.

1. Introduction

Plasmonics, the study of surface plasmons—coherent oscillations of free electrons at the interface between materials—has garnered significant attention due to its potential applications in subwavelength optics, sensing, and nanophotonic devices [1-3]. Surface plasmons enable the confinement of electromagnetic energy beyond the diffraction limit, facilitating the development of miniaturized optical components. In plasma physics, understanding the interaction between electromagnetic fields and charged particles is crucial. Traditional electrostatic models typically focus on non-relativistic phenomena, while relativistic effects can become important when addressing intense electromagnetic fields, such as laser-plasma interactions [4-6]. Incorporating relativistic considerations into electrostatic modeling enhances the accuracy of simulations and predictions, particularly in high-energy environments [7,8].

Recent studies have highlighted the significance of both relativistic effects and electrostatic interactions in plasmonic phenomena. Notably, the enhancement of harmonic generation in laser-irradiated grating targets has been attributed to relativistic surface plasmons, where the interplay between the relativistic properties of charged particles and electrostatic fields plays a crucial role [9]. Furthermore, covariant formulations have been effectively applied in the study of surface plasmon resonance at high-energy interfaces [10]. These findings emphasize the importance of considering both relativistic and electrostatic factors in advancing the understanding of plasmonic behaviors.

A covariant formulation of electrostatic models ensures that the equations governing plasma behavior remain consistent under Lorentz transformations, adhering to the principles of special relativity. This approach provides a unified framework for analyzing plasma phenomena across different inertial frames, which is essential for accurately describing systems involving high velocities or strong fields [11,12]. In this context, this paper introduces a local relativistic electrostatic model with a covariant formulation specifically designed for plasma physics applications. By integrating relativistic effects and ensuring covariance, the model offers a more accurate and comprehensive tool for analyzing plasma behaviors.

2. Electrostatic Model of Plasma

For a traditional electrostatic plasma, the dynamics of relativistic charged particles and the associated electric fields are governed by the Newton-Poisson equations:

$$\frac{d}{dt}(\gamma_a m_a \dot{\mathbf{X}}_a) = q_a \mathbf{E}, \#(1)$$

$$\nabla \cdot \mathbf{E} = 4\pi \sum_a q_a \delta(\mathbf{x} - \mathbf{X}_a), \#(2)$$

where $\mathbf{X}_a(t)$ represents the trajectory of the a -th particle, and q_a and m_a are particle charge and mass, respectively. The Lorentz factor of the a -th particle is given by $\gamma_a \equiv 1/\sqrt{1 - \dot{\mathbf{X}}_a^2/c^2}$, and the static electric field $\mathbf{E}(t, \mathbf{x}) = -\nabla\phi$ is a function of space and time, with ϕ being the electric potential. Equations (1) and (2) commonly employed in the study of electrostatic waves, Landau damping, and electrostatic turbulence in plasmas. However, these equations are not relativistically covariant due to the absence of magnetic fields. This non-covariance does not adhere to the universal principles of special relativity, necessitating their conversion into a relativistic framework. Under a Lorentz transformation with a relative velocity \mathbf{v} , the electric and magnetic fields transform as follows:

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \#(3)$$

$$\mathbf{E}'_{\perp} = \gamma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad \mathbf{B}'_{\perp} = \gamma \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} \right), \#(4)$$

where $\gamma \equiv 1/\sqrt{1 - \mathbf{v}^2/c^2}$. This implies that even if only an electric field exists in the rest frame, a magnetic field can appear in a series of different moving frames. In other words, electrostatics cannot be considered as the sole presence of an electric field; the existence of a magnetic field must also be considered in a relativistic context. For magnetostatics, the situation is analogous: even if only a magnetic field exists in the rest frame, an electric field can arise in various different moving frames. This suggests that we cannot distinguish between electrostatics and magnetostatics by relying solely on the presence of either the electric field or the magnetic field. Instead, we must consider the electric and magnetic fields as interconnected components or utilize the electro-magnetic tensor (or Faraday tensor) to differentiate between electrostatics and magnetostatics.

The electromagnetic tensor and its dual are expressed as follows:

$$F_{\mu\nu} = \begin{pmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & -\epsilon \cdot \mathbf{B} \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -\mathbf{E} \\ \mathbf{E} & -\epsilon \cdot \mathbf{B} \end{pmatrix}, \#(5)$$

$${}^*F_{\mu\nu} = \begin{pmatrix} 0 & -\mathbf{B} \\ \mathbf{B} & -\epsilon \cdot \mathbf{E} \end{pmatrix}, \quad {}^*F^{\mu\nu} = \begin{pmatrix} 0 & \mathbf{B} \\ -\mathbf{B} & -\epsilon \cdot \mathbf{E} \end{pmatrix}, \#(6)$$

where ϵ denotes the totally antisymmetric Levi-Civita tensor in 3D space, and ${}^*F_{\mu\nu}$ is the dual of $F_{\mu\nu}$ defined by ${}^*F_{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho}$ with $\epsilon^{0123} = -\epsilon_{0123} = 1$. An electrostatic system can be defined as one in which there exists an appropriate reference frame, or a local reference frame, where the magnetic field is zero. In such a case, the electromagnetic tensor simplifies to

$$F_{\mu\nu} = \begin{pmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & \mathbf{0} \end{pmatrix}, \#(7)$$

where $\mathbf{0}$ is the 3D zero tensor. This implies that we can find an observer with a 4-velocity $Z^\nu = \gamma(c, \mathbf{v})$ such that

$${}^*F_{\mu\nu} Z^\nu = 0. \#(8)$$

Conversely, a magnetostatic system can be defined as one in which we can always find an appropriate reference frame, or a local reference frame, where the electric field is zero. In this scenario, the dual electromagnetic tensor can be expressed as

$${}^*F_{\mu\nu} = \begin{pmatrix} 0 & -\mathbf{B} \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \#(9)$$

This means that we can find an observer whose 4-velocity is $Z^\nu = \gamma(c, \mathbf{v})$ such that

$$F_{\mu\nu}Z^\nu = 0. \#(10)$$

We can demonstrate (see Sec. 3) that it is fundamentally impossible to find two reference frames such that the magnetic field is zero in one and the electric field is zero in the other. In other words, the system cannot be both electrostatic and magnetostatic. Mathematically, this implies that we cannot spontaneously find a 4-velocity Z^ν that simultaneously satisfies equations (8) and (10).

3. Eletrostatics and Magnetostatics Classifications Using Electromagnetic Invariants

From the electric and magnetic field intensities, we can derive two invariant quantities that remain unchanged under transformations between inertial reference frames. These invariants can be expressed in terms of the electromagnetic tensor $F_{\mu\nu}$. Specifically, they are given by

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \mathbf{E}^2 - \mathbf{B}^2 = \text{inv}, \#(11)$$

$$\frac{1}{4}F_{\mu\nu} *F^{\mu\nu} = \mathbf{E} \cdot \mathbf{B} = \text{inv}. \#(12)$$

Equation (11) implies that if $\mathbf{E}^2 - \mathbf{B}^2 > 0$ in one reference frame, this inequality will hold true in all reference frames. We now demonstrate how to use these two invariants to classify electrostatic, magnetostatic, and electromagnetic systems. To begin, we present the following proposition:

3.1 Assume that $\mathbf{E} \cdot \mathbf{B} = 0$ and either $\mathbf{E} \neq 0$ or $\mathbf{B} \neq 0$. Then the following conditions hold

- (1) $|\mathbf{E}| > |\mathbf{B}|$, if and only if there exists a reference system in which the magnetic field vanishes, ($\mathbf{B}' = 0$);
- (2) $|\mathbf{B}| > |\mathbf{E}|$, if and only if there exists a reference system in which the electric field vanishes, ($\mathbf{E}' = 0$);
- (3) $|\mathbf{E}| = |\mathbf{B}|$, if and only if there is no coordinate system in which either the magnetic field or the electric field.

This proposition is proved as follows. Firstly, we prove the necessity of (1) and (2): If in a certain coordinate system, $\mathbf{B}' = 0$ or $\mathbf{E}' = 0$, then $\mathbf{E}' \cdot \mathbf{B}' = 0$. From Eq. (12), it is evident that in other inertial frames, even if the magnetic field (or electric field) is non-zero, the condition $\mathbf{E} \cdot \mathbf{B} = 0$ still holds, thereby confirming the necessity. Next, we demonstrate the sufficiency of (1) and (2), we first express Eqs. (1) and (2) in a more compact form

$$\mathbf{E}' = \gamma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - (\gamma - 1) \frac{(\mathbf{E} \cdot \mathbf{v})}{v^2} \mathbf{v}, \#(13)$$

$$\mathbf{B}' = \gamma \left(\mathbf{B} + \mathbf{E} \times \frac{\mathbf{v}}{c} \right) - (\gamma - 1) \frac{(\mathbf{B} \cdot \mathbf{v})}{v^2} \mathbf{v}. \#(14)$$

If we let the observer move along the direction of the electromagnetic field's energy flux density, the velocity satisfies

$$\mathbf{v} = ck\mathbf{E} \times \mathbf{B}, \#(15)$$

where k is a constant related to \mathbf{E} and \mathbf{B} , and $|\mathbf{v}| < c$. Substituting (15) into (13) and (14) yields

$$\mathbf{E}' = \gamma(1 - kB^2)\mathbf{E}, \#(16)$$

$$\mathbf{B}' = \gamma(1 - kE^2)\mathbf{B}. \#(17)$$

If $|\mathbf{E}| > |\mathbf{B}|$, we can choose $k = |\mathbf{E}|^{-2}$, then

$$\mathbf{v} = c\mathbf{E} \times \frac{\mathbf{B}}{|\mathbf{E}|^2} \#(18)$$

satisfying $|\mathbf{v}| < c$. From Eq. (17), it can be seen that $\mathbf{B}' = 0$; if $|\mathbf{B}| > |\mathbf{E}|$, we can choose ($k = |\mathbf{B}|^{-2}$), then

$$\mathbf{v} = c\mathbf{E} \times \frac{\mathbf{B}}{|\mathbf{B}|^2} \#(19)$$

satisfying $|\mathbf{v}| < c$.

From Eq. (16), it can be seen that $\mathbf{E}' = 0$; thus, we have proven the existence. But is there uniqueness? Not necessarily. Firstly, we note that if the velocity is along the energy flux direction, it must be determined as Eq. (18) or (19) to make the magnetic or electric field zero. Thus, assuming there is another solution, the velocity of the other observer cannot continue along the energy flux direction with respect to the original coordinate system. If this observer's velocity is not along the direction perpendicular to the electric field (or magnetic field), then the electric field (or magnetic field) must have a component parallel to this velocity direction. At this point, using Eq.(1), it can be seen that from the point of view of this observer, the electric field (or magnetic field) will not be zero. Therefore, if this observer observes the electric field (or magnetic field) to be zero, it must be moving along a direction perpendicular to the electric field (or magnetic field). Moreover, if $|\mathbf{E}| > |\mathbf{B}|$ (or $|\mathbf{B}| > |\mathbf{E}|$), we can only make the magnetic field (or electric field) zero; otherwise, it would violate Eq. (11). This means that if there exists a reference system in which the magnetic field is observed to be zero while the electric field is not, then there will never exist a coordinate system in which the electric field is observed to be zero, and similarly, if there exists a coordinate system in which the electric field is observed to be zero while the magnetic field is not, then there will never exist a coordinate system in which the magnetic field is observed to be zero.

We will now discuss the case where $|\mathbf{E}| > |\mathbf{B}|$. From the above discussion, we assume that the velocity must satisfy

$$\mathbf{v} = ck\mathbf{E} \times \mathbf{B} + \lambda\mathbf{E} \#(20)$$

Substituting Eq. (20) into (14), we have

$$\mathbf{B}' = \gamma[\mathbf{B} + \mathbf{E} \times (k\mathbf{E} \times \mathbf{B} + \lambda\mathbf{E})] = \gamma(1 - k|\mathbf{E}|^2)\mathbf{B} \#(21)$$

From Eq. (21), we can see that the value of \mathbf{B}' is independent of λ , as long as the velocity component along the energy flux direction satisfies $k = |\mathbf{E}|^{-2}$, meaning the velocity in that direction remains as Eq. (18). Thus, we conclude that when the velocity satisfies

$$\mathbf{v} = c \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E}|^2} + \lambda\mathbf{E}, \#(22)$$

the magnetic field can be zero. Similarly, for the case where $|\mathbf{B}| > |\mathbf{E}|$, when the velocity satisfies

$$\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{B}|^2} + \mu\mathbf{B}, \#(23)$$

the electric field can be made zero. Next, we prove (3). Given that $|\mathbf{E}| = |\mathbf{B}|$, and according to Eq. (11), if a coordinate system can be found where the magnetic field is zero, the electric field must also be zero, and conversely. As previously discussed, this necessitates the observer's velocity to be orthogonal to both the electric and magnetic fields. Therefore, the velocity direction must be along the energy flux (i.e., parallel to $\mathbf{E} \times \mathbf{B}$), meaning the observer's velocity satisfies

$$\mathbf{v} = cv\mathbf{E} \times \mathbf{B}. \#(24)$$

Substituting Eq. (24) into Eqs. (13) and (14), we obtain

$$\mathbf{E}' = \gamma(1 - vB^2)\mathbf{E}, \#(25)$$

$$\mathbf{B}' = \gamma(1 - vE^2)\mathbf{B}. \#(26)$$

If either the magnetic or electric field is zero, then from Eq. (13) or (14), we will obtain the same v that satisfies $v = |\mathbf{B}|^{-2} = |\mathbf{E}|^{-2}$. At this point, the observer's velocity is

$$\mathbf{v} = c \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{B}|^2} = c \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E}|^2}. \#(27)$$

However, since $|\mathbf{E}| = |\mathbf{B}|$, we have $|\mathbf{v}| = c$, the speed of light, which contradicts the principle of relativity. Therefore, such an observer does not exist, which means that there is no such reference frame that can make the electric or magnetic field zero.

From the proof of Proposition 1, we infer that if $\mathbf{E} \cdot \mathbf{B} = 0$ and $|\mathbf{E}| = |\mathbf{B}|$, then to nullify both the electric and magnetic fields, the observer's speed would need to be the speed of light, which is not possible for any observer. If $|\mathbf{E}| \neq |\mathbf{B}|$, Eq.(13), indicates that $|\mathbf{E}|^2 - |\mathbf{B}|^2 \neq 0$ in any coordinate system, implying that there is no reference system where $|\mathbf{E}| = |\mathbf{B}| = 0$.

3.2 The condition $\mathbf{E} \cdot \mathbf{B} \neq 0$ holds true if and only if it is impossible to find a coordinate system where both the electric field and the magnetic field are zero.

The proof of this proposition is clear when using Eq. (12). If $\mathbf{E} \cdot \mathbf{B} \neq 0$, there exists no reference frame in which both the electric and magnetic field are zero, i.e., $|\mathbf{E}| = |\mathbf{B}| = 0$.

From the discussion above, we have established that for an electromagnetic field to be classified as electrostatic or magnetostatic, it must fulfill the condition of $\mathbf{E} \cdot \mathbf{B} = 0$. Furthermore, it is evident that there is no overlap between the domains of electrostatic and magnetostatic fields. Consequently, we can derive an equivalent definition for the fields of electrostatics, magnetostatics, and electromagnetism, as presented in Table I below.

Table 1. Classifications for different electromagnetic states by invariants

Classifications	Invariants	Covariant Form
Electrostatic	$\mathbf{E} \cdot \mathbf{B} = 0$ and $\mathbf{E}^2 - \mathbf{B}^2 > 0$	$F_{\mu\nu}F^{\mu\nu*} = 0$ and $F_{\mu\nu}F^{\mu\nu} < 0$
Magnetostatic	$\mathbf{E} \cdot \mathbf{B} = 0$ and $\mathbf{E}^2 - \mathbf{B}^2 < 0$	$F_{\mu\nu}F^{\mu\nu*} = 0$ and $F_{\mu\nu}F^{\mu\nu} > 0$
Electromagnetic	$\mathbf{E} \cdot \mathbf{B} \neq 0$ or $\mathbf{E}^2 - \mathbf{B}^2 = 0$	$F_{\mu\nu}F^{\mu\nu*} \neq 0$ or $F_{\mu\nu}F^{\mu\nu} = 0$

4. Local Relativistic Electrostatic Equations

Based on the preceding discussion, it becomes readily apparent that the definitions provided above are equivalent to those we initially presented in Sec. 2, at least in a local sense. To illustrate this point, let us take the example of the electrostatic electromagnetic field. The reason for the locality is evident when examining Eq. (18), which reveals that the velocity of our observer must be a function of the electromagnetic field. This implies that if the electromagnetic field is nonuniform or time varying, the observer's velocity must differ at various spacetime locations. Consequently, the reference frame we choose appears akin to a "soft-bodied creature" that is perpetually "squirming" in three-dimensional space, with its elements consisting of "instantaneous inertial" observers strategically positioned at arbitrary spacetime points. For any given "instantaneous inertial" observer, within their local vicinity, if their velocity relative to the original coordinate system follows the prescription of Eq. (18), they will "locally" observe a magnetic field of zero while the electric field remains non-zero.

To ensure that equations Eq. (1) and (2) are covariant, it is necessary to take the magnetic field into account. To maintain the system in an electrostatic state, we must fulfill the conditions corresponding to the second row in Table I. The complete covariant form of the local electrostatic equations for the plasma should satisfy the following equations:

$$\frac{d}{dt}(\gamma_a m_a \dot{\mathbf{X}}_a) = q_a \left(\mathbf{E} + \dot{\mathbf{X}}_a \times \frac{\mathbf{B}}{c} \right), \#(28)$$

$$\nabla \cdot \mathbf{E} = 4\pi \sum_a q_a \delta(\mathbf{x} - \mathbf{X}_a), \#(29)$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \sum_a q_a \dot{\mathbf{X}}_a \delta(\mathbf{x} - \mathbf{X}_a), \#(30)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \#(31)$$

$$\mathbf{E} \cdot \mathbf{B} = 0, \#(32)$$

$$\mathbf{E}^2 - \mathbf{B}^2 > 0. \#(33)$$

The last two equations are included to guarantee that the system is locally electrostatic. Equations (28)-(33) can be recast in a manifestly covariant form as follows

$$\frac{d}{d\tau} (m_a \dot{X}_a^\mu) = q_a F_{\nu}^{\mu} \dot{X}_a^\nu, \#(34)$$

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} J^\nu, \#(35)$$

$$\partial_\mu {}^*F_{\mu\nu} = 0, \#(36)$$

$$F_{\mu\nu} {}^*F_{\mu\nu} = 0, \#(37)$$

$$F_{\mu\nu} F^{\mu\nu} > 0, \#(38)$$

where $\dot{X}_a^\mu = \gamma_a(c, \dot{\mathbf{X}}_a)$ denotes the four-velocity of the a -th particle, and $J^\nu \equiv \sum_a q_a \int \dot{X}_a^\nu \delta(x^\sigma - X_a^\sigma) dt$ represents the four-current. Using the 4-potential $A_\mu = (c\phi, \mathbf{A})$, equations (37) and (38) can be rewritten as

$$\frac{1}{2} F_{\mu\nu} {}^*F_{\mu\nu} = \partial_\mu ({}^*F_{\mu\nu} A_\nu) = 0, \#(39)$$

$$\partial_\mu (F^{\mu\nu} A_\nu) + 4\pi A^\nu J_\nu < 0, \#(40)$$

with Eq. (35) and (36) applied. Equation (39) enshrines the conservation of helicity in plasma, indicating that an electrostatic system must conserve helicity. Consequently, equations (34)-(38) can be reformulated as

$$\frac{d}{d\tau} (m_a \dot{X}_a^\mu) = q_a F_{\nu}^{\mu} \dot{X}_a^\nu, \#(41)$$

$$\partial_\mu \partial^\mu A^\nu = -\frac{4\pi}{c} J^\nu, \#(42)$$

$$\partial_\mu ({}^*F_{\mu\nu} A_\nu) = 0, \#(43)$$

$$\partial_\mu (F^{\mu\nu} A_\nu) + 4\pi A^\nu J_\nu < 0, \#(44)$$

where the Lorentz gauge $\partial_\mu A^\mu = 0$ is adopted. Equations (41)-(44) will henceforth be referred to as the manifestly covariant local electrostatic equations in plasmas.

5. Conclusions

This study underscores the importance of integrating relativistic considerations into electrostatic modeling to advance the understanding of plasmonic phenomena. By developing a local relativistic electrostatic model with a covariant formulation, we established a more accurate and comprehensive approach for analyzing plasma behavior in high-energy contexts. The findings reveal that accounting for both relativistic effects and electrostatic interactions is critical for understanding complex systems, particularly in laser-plasma scenarios. As the field of plasmonics continues to evolve, the methodologies and insights provided by this research pave the way for further exploration of surface plasmons and their applications in advanced optical technologies.

Future work should focus on refining these models and extending their applications to a wider range of physical phenomena.

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