

Research on Singularity in Calculation of Induced Electric Field in Pointed Conductors

Kejie Li^{1, a}, Dian Sun^{1, b, *}

¹ School of Electrical and Automation Engineering, Hefei University of Technology, Hefei 230009, China.

^a kejie.li@hotmail.com, ^{b,*} sundian@mail.hfut.edu.cn

Abstract. In the electric field environment of thunderstorm, the tip of a grounded object at a high place is more likely to produce corona discharge and finally cause lightning flash. Therefore, the study of the surface electric field of the grounded object tip in the electric field environment has important theoretical value and practical significance for the analysis of the characteristics of the head-on discharge. Due to the strong singularity of the tip structure, the traditional numerical analysis method cannot accurately calculate its surface electric field. Therefore, the traditional literature usually chamfers the conductor tip, but in lightning physics, researchers are most concerned about the maximum field strength in the local area. In order to study the electric field distribution on the tip conductor surface, a semi-analytical boundary element method is proposed to calculate the tip structure directly. Firstly, the analytical formula of the electric field on the tip conductor surface and the boundary element semi-analytical common method are derived, and the tip conductor model is established to calculate the tip electric field excited by the external electric field using the analytical solution and the semi-analytical method. The correctness of the semi-analytical method is verified by calculating the L_2 loss function of the two, and the influence of the tip angle on the semi-analytical calculation results is studied. The results show that for the three-dimensional cone model, the L_2 loss function is 10^{-5} , which meets the calculation accuracy. The special treatment of the semi-analytical method at the strong singular integral makes the calculation accuracy much higher than that of the Gaussian integral, where the L_2 loss function is greatly reduced. Further, based on the change of tip angle, as the conductor tip becomes sharper, the greater the electric field distortion is, and the greater the L_2 loss function is. The research results provide a method reference for the calculation of the electric field on the tip conductor surface.

Keywords: Tip conductor; electric field distribution; boundary element method; semi-analytical method; corona discharge.

1. Introduction

Under the influence of atmospheric electric fields, opposite polarity charges are induced near the ground. Due to the presence of tips, trees, insulated buildings, communication poles, and other protrusions on the ground can cause electric field distortion. If the electric field is strong enough, corona discharge, also known as tip discharge, will occur. Due to the fact that in the same lightning environment, the metal conductors at the end of the structure accumulate more heteropolar charges than non-metallic conductors, and face-to-face discharge often starts from the surface of the metal conductors at the end of the ground structure, with the starting characteristics determined by the induced electric field on the surface of the conductors. Therefore, the analysis of head-on discharge characteristics needs to be based on accurate calculation of the surface electric field of conductors at the end of ground structures. The study of surface electric field calculation is a fundamental work of great significance for the analysis of head-on discharge processes.

At present, the numerical methods for calculating 3D electric field mainly include finite difference method, finite element method, analog charge method and boundary element method. In order to accurately assess the lightning background field ground structures end of the induction charge and electric field, assessment of ground structure lightning risk or analysis of lightning development physical mechanism, many scholars use the mentioned numerical methods to calculate the cloud and the forerunner space field, especially the electric field distribution on the surface of

the end of the ground structure[8][9][10]. However, due to the aggregation effect of charge, there will be a "lightning rod effect" near the sharp structure, that is, the charge is concentrated on the surface of small curvature radius such as ends, corners, lightning rod and so on, etc., which makes the conventional numerical calculation method is easy to appear large error.

It is generally believed that the singularity of its local region will not affect the overall situation, but in lightning physics, the researchers are most concerned about is the local area of the maximum field strength. No matter in the pilot starting stage or the development stage, the physical process of lightning is always closely related to the maximum field strength in the space. Although the electric field calculation error at the local point does not affect the distribution characteristics of the field or the integration along the line, it directly affects a series of problems such as the starting position, starting criterion and development direction of the pilot.

In this paper, we use the tip as the conductor object and use semi analytical methods to solve indirect boundary element equations for calculating surface charges and induced electric fields of tip structures by the MATLAB program, a semi-analytical method was proposed to improve the calculation accuracy of strong singular integrals, and analyzed and the influence of the change of the tip angle on the calculation accuracy.

2. Derivation of indirect boundary element analytical solution formula

2.1 The basic principle

When a conductor in an electrostatic field, under the action of the electric field, can make the charge of the conductor surface moving, at the same time on the conductor because of the external electric field and moving charge, its itself will form an electric field, the electric field and superposition to the original background electric field, makes the electric field around the surface of the conductor has changed.

In three dimensions, the potential φ satisfies the Poisson equation:

$$\nabla^2 \varphi = -\frac{\tau}{\varepsilon} \quad (1)$$

Where: τ is the density of the charge surfaces of the space.

Let's assume that the Green's function $G(\vec{r}, \vec{r}')$ satisfies the equation:

$$\nabla^2 G(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}') \quad (2)$$

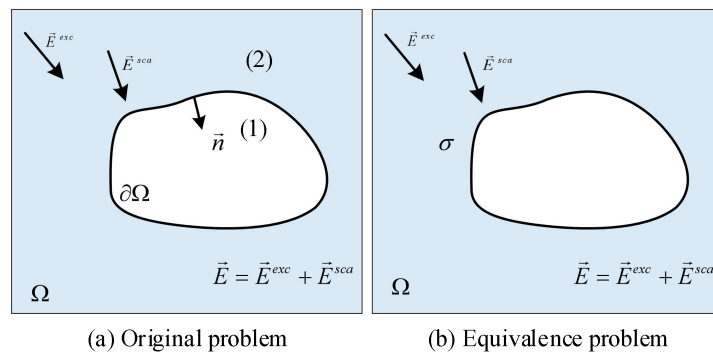


Figure. 1 The conductor induces the surface charge under the action of the excitation field to form the induction field

Since the internal charge of the conductor reaches zero after reaching electrostatic equilibrium, the induced field is only caused by the surface charge of the conductor. It can be assumed that the conductor is withdrawn and only the induced charge in the original position, when the potential and spatial electric field remain unchanged. At this time, the solution domain becomes an infinite free space, with

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|} \quad (3)$$

The expression of potential can be obtained by using Green's function:

$$\varphi(r) = \frac{1}{4\pi\epsilon} \iint_{S'} \tau(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} dS' \quad (4)$$

Where: r is the field point position vector; r' is the source point position vector; τ is the charge density of the source point surface; φ is the field point potential; ϵ is the medium constant of vacuum; S is the area division of the source point.

The boundary integral equation is discretized using the point fitting method to obtain the point fitting boundary element equation[5]:

$$\varphi^{ind}(\vec{r}_j) = \frac{1}{4\pi\epsilon} \sum_{i=1}^N \iint_{S_i} \frac{1}{|\vec{r} - \vec{r}'|} dS_i \cdot \tau_i \quad (5)$$

Where: N is the number of surface cells; j is the number of the field cell; i is the number of the source cell; S_i represents the integration surface of the source element; τ the node surface charge density; φ^{ind} represents the node induced potential.

Assuming the external excitation is a uniform electric field $\vec{E}^{exc}$ with arbitrary direction, and the potential in the external field alone is:

$$\varphi^{exc}(\vec{r}) = -\int_L \vec{E}^{exc} \cdot d\vec{l} + C \quad (6)$$

Where: L is any integral path from r to infinity. For uniform fields, φ^{exc} is a linear function along the direction of the electric field. The excitation field and the induction field together constitute the spatial potential $\varphi = \varphi^{exc} + \varphi^{ind}$. When the conductor is grounded $\varphi = 0$, then $\varphi^{ind} = -\varphi^{exc}$.

Let the vector $b = [\varphi_1^{ind}, \varphi_1^{ind} \dots \varphi_N^{ind}]^T$, $\tau = [\tau_1, \tau_2 \dots \tau_N]^T$, and write the node discrete variables in equation (5) and equation (6) in matrix form, so that it can be written as the following equation:

$$A_{ji} \tau = b, \text{ with } A_{ji} = \frac{1}{4\pi\epsilon} \iint_{S_i} \frac{1}{|\vec{r} - \vec{r}'|} dS_i \quad (7)$$

Solving the above equation obtains the line charge density τ of the conductor surface, and then the electric field strength of the conductor surface can be obtained.

2.2 Conical tip electric field expression.

Analyze the surface electric field intensity of a conical metal tip with a half vertex angle β and establish a coordinate system[3]. Let the potential on the conductor be 0, and the potential in the outer space satisfies the Laplace equation $\nabla^2 \varphi = 0$, when $r \rightarrow 0$, $\varphi(r, \theta)$ is a finite value, so the general solution of the Laplace equation in the axisymmetric potential problem is taken as

$$\varphi(r, \theta) = \sum_v A_v r^v P_v(\cos \theta) \quad (8)$$

Where: $P(x)$ ($x = \cos \theta$) satisfy Legendre's equation $(1-x^2) \frac{d^2 P}{dx^2} - 2x \frac{dP}{dx} + v(v+1)P = 0$.

Since we require that $P(x)$ is a regular function when $x=1$, it is convenient to expand $P(x)$ by series with $x=1$ and introduce the variables $\xi = (1-x)/2$. Substitute the power series solution

$P(\xi) = \xi^\alpha \sum_{j=0}^{\infty} a_j \xi^j$ in, as the coefficient of the lowest power of ξ is equal to zero, this requires $\alpha=0$. Derived as

$$P_v(\xi) = 1 + \frac{(-v)(v+1)}{1!!} \xi + \frac{(-v)(-v+1)(v+1)(v+2)}{2!2!} \xi^2 + \dots \quad (9)$$

Because the potential at the origin is limited, the equation $v > 0$. In order to satisfy the boundary condition, when $\theta = \beta$, for all r , the potential must be equal to zero, so there must be

$$P_v(\cos \beta) = 0 \quad (10)$$

This is the v of the eigenvalue condition. Within the interval $0 \leq \theta \leq \beta$, the complete solution of the axisymmetric potential is

$$\varphi(r, \theta) = \sum_{k=1}^{\infty} A_k r^{v_k} P_{v_k}(\cos \theta) \quad (11)$$

The components of the surface electric field strength and surface charge density on the conical conductor are

$$E_r = -\frac{\partial \varphi}{\partial r} = -\sum_{k=1}^{\infty} v_k A_k r^{v_k-1} P_{v_k}(\cos \theta) \quad (12)$$

$$E_{\theta} = -\frac{1}{r} \frac{\partial \varphi}{\partial \theta} = -\sum_{k=1}^{\infty} A_k r^{v_k-1} \sin \theta P_{v_k}'(\cos \theta) \quad (13)$$

$$\sigma(\rho) = -\frac{\varepsilon}{4\pi} E_{\theta} |_{\theta=\beta} = -\frac{\varepsilon}{4\pi} \sum_{k=1}^{\infty} A_k r^{v_k-1} \sin \beta P_{v_k}'(\cos \beta) \quad (14)$$

It is not difficult to find that both the charge density and the electric field strength near the tip surface of the tip conductor are approximately variable with r^{v-1} near $r \rightarrow 0$, the charge density increases with the curvature of the conductor surface.

3. Semi-analytic method

In general, the Gaussian integration method is used to solve the boundary integral equation, but when the field point is close to the source point, the integrand function changes dramatically, especially when the field point and source point coincide, the integrand function is already a singular function. And the calculation accuracy of these parts is particularly important for the final accuracy of the entire boundary element method, so some special processing techniques need to be adopted. This article splits the integral equation and uses numerical integration methods to solve the parts without singular points; For the part containing singular points, the analytical expression is obtained by simplifying the integrand function through integral transformation[4], and the correctness of this method is verified through the calculation example.

This paper uses linear isoreference finite element units and 3-node triangular units. The shape function of the boundary surface element is $N_1 = \lambda_1, N_2 = \lambda_2, N_3 = 1 - \lambda_1 - \lambda_2$, formula (5) transformed into

$$\varphi = \frac{\Delta}{2\pi\varepsilon_0} \sum_{i=1}^m \int_0^1 \int_0^{1-\lambda_2} \frac{N_p(\lambda_1, \lambda_2)}{R(\lambda_1, \lambda_2)} d\lambda_1 d\lambda_2 \quad (15)$$

If the source triangle formula (17) has no singularity, the Gaussian integral can be used; When the field point is inside the source triangle, i.e. $R \rightarrow 0$, there is singularity in the integration. Let $F_p(\lambda_1', \lambda_2') = N_p$, $F_p(\lambda_1', \lambda_2')$ will be Taylor expansion at the point $(\lambda_{1,j}, \lambda_{2,j})$, only take one item

$$F_p(\lambda_1', \lambda_2') \approx F_p(\lambda_{1,j}, \lambda_{2,j}) + (\lambda_1' - \lambda_{1,j}) \frac{\partial F_p}{\partial \lambda_1'}(\lambda_{1,j}, \lambda_{2,j}) + (\lambda_2' - \lambda_{2,j}) \frac{\partial F_p}{\partial \lambda_2'}(\lambda_{1,j}, \lambda_{2,j}) \quad (16)$$

Substituting formula (18) into formula (17) yields an expression that only contains three singular

integral terms, $\int_0^1 \int_0^{1-\lambda_2'} \frac{1}{R(\lambda_1', \lambda_2')} d\lambda_1' d\lambda_2'$, $\int_0^1 \int_0^{1-\lambda_2'} \frac{\lambda_1' - \lambda_{1,j}}{R(\lambda_1', \lambda_2')} d\lambda_1' d\lambda_2'$ and $\int_0^1 \int_0^{1-\lambda_2'} \frac{\lambda_2' - \lambda_{2,j}}{R(\lambda_1', \lambda_2')} d\lambda_1' d\lambda_2'$. the three basic integrals are computed analytically below.

The field points are the vertices of the triangular element, so that the singularity is always at an angle of the triangle. We discuss it in three cases, namely, point P_0 is at point P_1, P_2, P_3 of the triangular element.

If $\begin{cases} u = \lambda'_1 - \lambda_{1,j} \\ v = \lambda'_2 - \lambda_{2,j} \end{cases}$, i.e. $\int_0^1 \int_0^{1-\lambda'_2} d\lambda'_1 d\lambda'_2 = \int_{-\lambda_{2,j}}^{1-\lambda_{2,j}} \int_{-\lambda_{1,j}}^{1-\lambda_{1,j}-\lambda_{2,j}} dudv$. Take $P_0 = P_1$ for example, $\lambda_{1,j} = 0, \lambda_{2,j} = 0$.

$$\int_0^1 \int_0^{1-\lambda'_2} \frac{1}{R(\lambda'_1, \lambda'_2)} d\lambda'_1 d\lambda'_2 = \int_0^1 \int_{-1}^{-v} \frac{1}{R(u, v)} dudv = \frac{1}{\sqrt{c}} \ln \frac{\sqrt{c}\sqrt{a-2b+c} - b + c}{\sqrt{a}\sqrt{c} - b} \quad (17)$$

$$\int_0^1 \int_0^{1-\lambda'_2} \frac{\lambda'_1 - \lambda_{1,j}}{R(\lambda'_1, \lambda'_2)} d\lambda'_1 d\lambda'_2 = \int_0^1 \int_{-1}^{-v} \frac{u}{R(u, v)} dudv = \frac{1}{2\sqrt{c}} \ln \frac{\sqrt{a}\sqrt{c} + b}{\sqrt{c}\sqrt{a-2b+c} + b - c} \quad (18)$$

$$\int_0^1 \int_0^{1-\lambda'_2} \frac{\lambda'_2 - \lambda_{2,j}}{R(\lambda'_1, \lambda'_2)} d\lambda'_1 d\lambda'_2 = \int_0^1 \int_{-1}^{-v} \frac{v}{R(u, v)} dudv = \frac{\sqrt{a-2b+c} - \sqrt{a}}{2c} + \frac{b}{2c^{3/2}} \ln \frac{\sqrt{a}\sqrt{c} + b}{\sqrt{c}\sqrt{a-2b+c} + b - c} \quad (19)$$

The integral when $P_0 = P_2$ and $P_0 = P_3$ can be obtained by the same way.

4. Numerical simulation results

To simulate the atmospheric electric field in lightning environments, it is assumed that there is a background electric field with a magnitude of 50kV/m pointing upwards. Firstly, the surface of the tip is dissected using COMSOL finite element analysis software. Then, the known ground zero potential is used as the first type of boundary condition, and the surface charge density of the tip is calculated using MATLAB semi analytical algorithm and analytical formula, respectively. On this basis, the height of the fixed conductor remains unchanged, and the influence of the tip angle on the numerical calculation results is analyzed.

The analytical solution formulas for the tip surface charge density have been provided in equation (14) in the previous text. For open region problems, it is necessary to set potential boundary conditions infinitely far from the tip. This article assumes that $\varphi(\rho, \theta) = -E(1 + \rho \cos \theta)$ with $\rho = 10000$, and calculate to the fifth term of the series, i.e k=5.

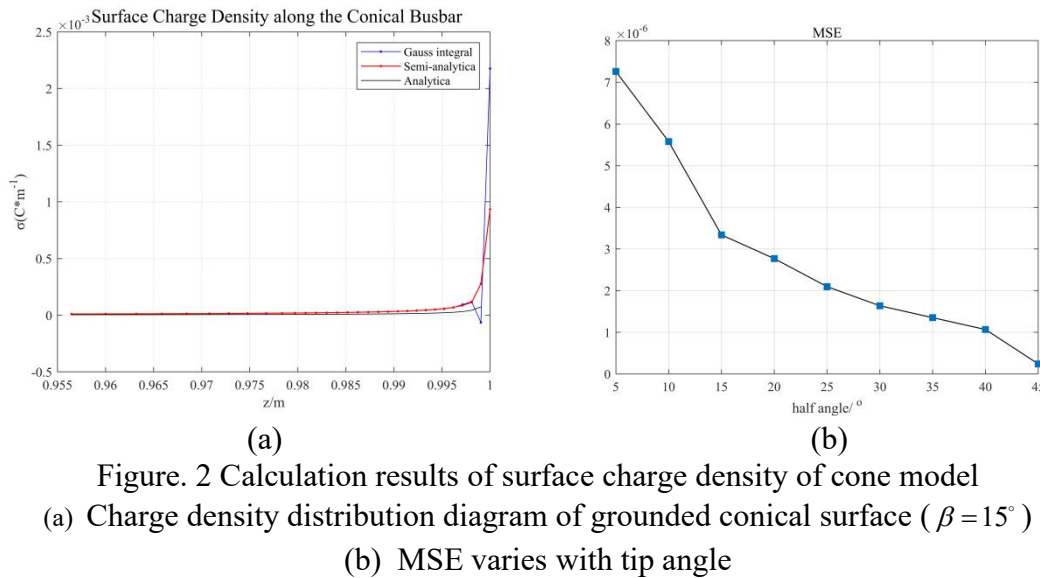


Figure. 2 Calculation results of surface charge density of cone model

(a) Charge density distribution diagram of grounded conical surface ($\beta = 15^\circ$)

(b) MSE varies with tip angle

Taking the $\beta = 15^\circ$ grounding cone as an example, calculate the surface charge density at the tip using ordinary Gaussian integration method, semi analytical strong singular integration processing method, and analytical formula, respectively. Figure 2(a) shows the distribution of surface charge density along the conical generatrix direction at the tip. From the graph, we can see that the closer it is to the tip, the higher its surface charge density, which increases exponentially. The surface charge density is highest at the tip, and the analytical solution tends towards infinity. The ordinary Gaussian integration method results in negative growth near the tip due to the presence of strong singular integrals, but our proposed semi-analytical method effectively solves this problem. The calculation results of the semi analytical algorithm and Gaussian integration method compare with

the analytical solution, the L2 mean square error (MSE) is $7.4981\text{e-}03$ and $3.2776\text{e-}06$, respectively, which greatly improves the computational accuracy of the semi analytical method. At the same time, this article analyzed the computational accuracy of the semi analytical method under different tip angles, as shown in Figure 2(b). As the tip angle continues to increase, the L2 mean square error gradually decreases.

Summary

5. Summary

This article uses the boundary element method to calculate the surface field strength of conductor tips in lightning environments, and proposes a semi analytical method to solve the problem of strong singular integration. Based on the analytical solution, compared with the Gaussian integration method, this method has significantly improved solution accuracy and is more suitable for solving the surface field strength at the tip structure. It is of great significance for numerical simulation of lightning discharge physical processes and evaluation of lightning protection measures for ground structures. Meanwhile, we found that as the blade tip angle decreases, the calculation error will relatively increase. Based on this discovery, we will continue to improve our methods and enhance computational accuracy.

Acknowledgment

Project supported by the Science and Technology Project of State Grid Corporation of China (5500 - 202155584A - 0 - 5 - SF).

References

- [1] Chen Shixiu, Shen Yuanmao, Wu Yuanli, et al. An equivalent model for electromagnetic waves radiated by needle plate electrode corona discharge [J]. Gao Dianshu, 2002 (S1): 9-11. DOI: 10.13336/j.1003-6520.hve.2002.s1.05
- [2] Luo Hua. Research on corona discharge characteristics of metal tips [D]. Nanjing University of Information Technology, 2021. DOI: 10.27248/d.cnki.gnjqc.2021.000129
- [3] Zhou Xuehui, Chen Dengyi, Chen Zehuang. Research on the threshold of metal tip corona discharge under atmospheric electric field [J]. Electric porcelain lightning arrester, 2016, No.273 (05): 152-159. DOI: 10.16188/j.isa.1003-8337.2016.05.029
- [4] Li Yasha, Wang Zezhong, Lu Binxian. Analytical integration method in three-dimensional electrostatic field linear interpolation boundary element method [J]. Computational Physics, 2007, (01): 59-64. DOI: 10.19596/j.cnki.1001-246x.2007.01.010
- [5] Jin Jianming, Wang Jianguo, Ge Debiao. Finite Element Method for Electromagnetic Fields. Xi'an: Xi'an Electronic Technology University Press, 1988
- [6] A. B. A. Halim, Z. A. Alyaa, I. H. Azil, et al. Determination of the striking distance of a lightning rod using finite element analysis[J]. Turkish Journal of Electrical Engineering and Computer Sciences, 2016, 24(5):4083-4097.
- [7] K. Bhattacharya. On the Dependence of Charge Density on Surface Curvature of an Isolated Conductor[J]. Physica Scripta, 2016, 91(3):035501
- [8] Y. Baba, V. A. Rakov. FDTD simulation of field-reduction effect at ground due to corona at lightning-triggering wire[C]// General Assembly & Scientific Symposium, IEEE, 2011:1-4
- [9] C. Zhuang, Y. Zhang, R. Zeng. Electric field evaluation around slender conductors by collocation boundary element method[J]. CSEE Journal of Power and Energy Systems, 2019.
- [10] Y. Zhang, C. Zhuang and R. Zeng, "Electrical field evaluation around slender conductors by collocation boundary element method," 2016 IEEE Conference on Electromagnetic Field Computation (CEFC), Miami, FL, USA, 2016, pp. 1-1, doi: 10.1109/CEFC.2016.7816125.

- [11] Z. Yu, L. Qian, Z. Rong, et al. Calculation of surface electric field on UHV transmission lines under lightning stroke[J]. Electric Power Systems Research, 2013, 94(Jan.):79-85.
- [12] D. Poljak, M. Birkic, D. Kosor, et al. Electromagnetic modeling of a lightning rod[J]. Wit Transactions on Modelling & Simulation, 2007.